

THE POINCARÉ HOMOLOGY SPHERE, LENS SPACE SURGERIES, AND SOME KNOTS WITH TUNNEL NUMBER TWO.

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ABSTRACT. We exhibit an infinite family of knots in the Poincaré homology sphere with tunnel number 2 that have a lens space surgery. Notably, these knots are not doubly primitive and provide counterexamples to a few conjectures. In the appendix, it is shown that hyperbolic knots in the Poincaré homology sphere with a lens space surgery has either no symmetries or just a single strong involution.

1. A FAMILY OF KNOTS IN THE POINCARÉ HOMOLOGY SPHERE WITH LENS SPACE SURGERIES.

Figure 1 shows a one-parameter family of arcs κ_n , $n \in \mathbb{Z}$, on the pretzel knot $P(-2, 3, 5) = P(3, 5, -2)$ for which a -1 -framed banding produces the two bridge knot $B(3n^2 + n + 1, -3n + 2)$. Indeed, $[n, -1, -n, 3] =$

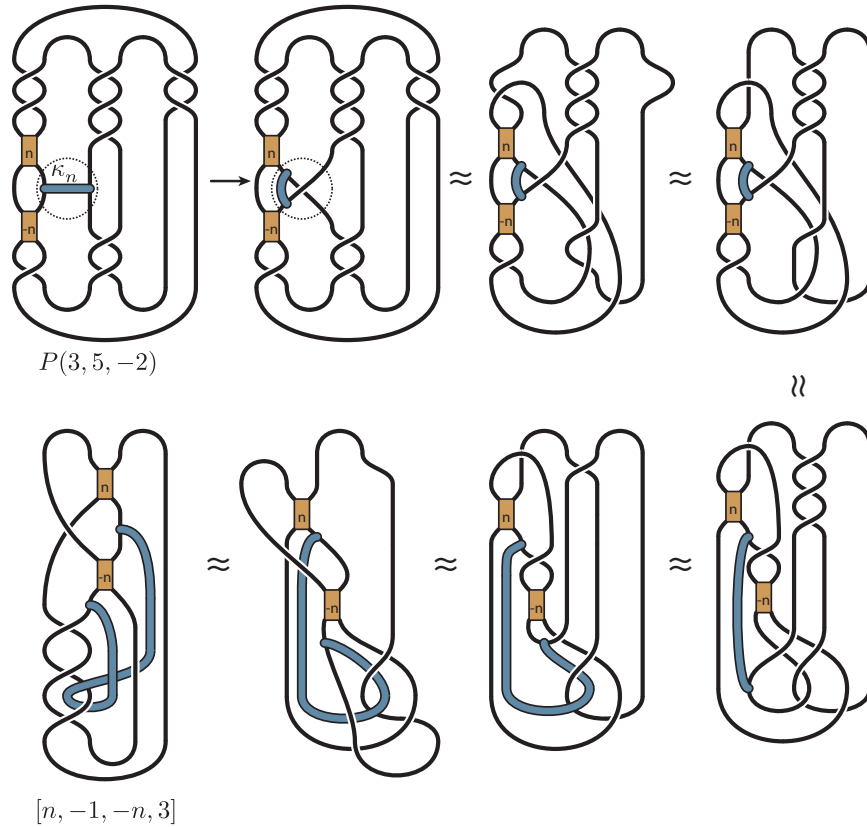


FIGURE 1. Top left shows a family of arcs κ_n with end points on the pretzel knot $P(-2, 3, 5)$. A numbered vertical box signifies a stack of half-twists where the sign informs the handedness. We perform a banding on κ_n (framed relative to the horizontal bridge sphere) followed by a sequence of isotopies to recognized the result as the two-bridge knot $B(3n^2 + n + 1, -3n + 2)$. The arc κ_n^* dual to the banding is also carried along.

$-\frac{3n^2+n+1}{-3n+2}$. Through double branched coverings and the Montesinos Trick, the arc κ_n lifts to a knot K_n in the Poincaré homology sphere on which an integral surgery yields the lens space $L(3n^2 + n + 1, -3n + 2)$.

As presented, the arc κ_n lies in a bridge sphere that presents $P(-2, 3, 5)$ as a 3-bridge link. Thus the knot K_n lies in a genus 2 Heegaard surface for the Poincaré homology sphere \mathcal{P} . Nevertheless, as we shall see, generically K_n does not have tunnel number one.

Theorem 1. *For each integer n , the knot K_n in the Poincaré homology sphere \mathcal{P} has an integral surgery to the lens space $L(3n^2 + n + 1, -3n + 2)$. If $|n| \geq 4$, then K_n has tunnel number 2.*

Previously, the only known knots in \mathcal{P} admitting a surgery to a lens space are surgery duals to Tange knots [Tan09a] and certain Hedden knots [Hed11, Ras07, Bak14b]. (See Section 5 for a discussion of the Hedden knots.) These knots in \mathcal{P} are all *doubly primitive*, they may be presented as curves in a genus 2 Heegaard splitting that represent a generator in π_1 of each handlebody bounded by the Heegaard surface. We see this as Tange knots and Hedden knots all admit descriptions by doubly pointed genus 1 Heegaard diagrams. (Tange knots are all simple knots in lens spaces and Hedden knots are “almost simple”.) Equivalently, they are all 1-bridge knots in their lens spaces. Any integral surgery on a 1-bridge knot in a lens space naturally produces a 3-manifold with a genus 2 Heegaard surface in which the dual knot sits as a doubly primitive knot.

The Berge Conjecture posits that any knot in S^3 with a lens space surgery is a doubly primitive knot [Ber], cf. [Gor91, Question 5.5]. It is also proposed that any knot in $S^1 \times S^2$ with a lens space surgery is a doubly primitive knot [BBL13, Conjecture 1.1], [Gre13, Conjecture 1.9]. By analogy and due to a sense of simplicity¹, one may suspect that any knot in \mathcal{P} with a lens space surgery should also be doubly primitive. However this is not the case. Since our knots K_n have tunnel number 2 for $|n| \geq 4$, they cannot be doubly primitive.

Corollary 2. *There are knots in the Poincaré homology sphere with lens space surgeries that are not doubly primitive. That is, the analogy to the Berge Conjecture fails for the Poincaré homology sphere.* \square

Corollary 3. *Conjecture 1.10 of [Gre13] is false. For example, the lens space $L(5, 1)$ contains a knot K_1^* with a surgery to \mathcal{P} but does not contain a Tange knot or a Hedden knot with a \mathcal{P} surgery.*

Proof. For $n = 1$, surgery on K_1 produces $L(5, -1)$ as shown in Figure 1. A quick check of [Tan09a, Table 2] shows that $L(5, -1)$ contains no Tange knots. The Hedden knots with surgery to $\pm\mathcal{P}$ are homologous to Berge lens space knots of type VII [Ras07, Gre13]; one further observes that $L(5, -1)$ does not contain any of these knots since $k^2 + k + 1 = 0 \pmod{5}$ has no integral solution. \square

Remark 4. The knot K_1 is actually the $(2, 3)$ -cable of the knot J that is surgery dual to the trefoil in S^3 .

As we shall see in Theorem 7(3), it is the $(2g(K_n) - 1)$ -surgery on our knots K_n that produces a lens space; $g(K_n)$ is the Seifert genus of the knot K_n . Hence they also provide counterexamples to [Hed11, Conjecture 1.7] (cf. [BGH08, Conjecture 1.6]) and [Gre13, Conjecture 1.10]. It seems plausible that it is the largeness of the knot genus with respect to the lens space surgery slope that enables the failure of our knots to be doubly primitive. Hedden and Rasmussen also observe a distinction at this slope for lens space surgeries on knots in L-space homology spheres [Hed11, Ras07]. Indeed, from the view of Heegaard Floer homology for integral slopes on knots in homology spheres, this slope is right at the threshold at which a knot could have an L-space surgery [OS11, Proposition 9.6] (see also [Hed09, Lemma 2.13])² and just below what implies it has simple knot Floer homology [Eft11]³. With this in mind, we adjust and update [Gre13, Conjecture 1.10].

Conjecture 5. *Suppose that p -surgery on a knot K in the Poincaré homology sphere produces a lens space. If $p > 2g(K) - 1$ is an integer, then K is a doubly primitive knot. Furthermore it is surgery dual to one of the Tange knots.*

Conjecture 6. *If a knot in the Poincaré homology sphere is doubly primitive, then it is surgery dual to one of the Tange knots or one of the Hedden knots.*

¹The manifolds S^3 , \mathcal{P} , and its mirror are the only homology 3-spheres with finite fundamental group (by Perelman) and the only known irreducible L-space homology 3-spheres, e.g. [Eft09, Eft15].

²This threshold may be lower for knots in homology spheres with $\tau(K) < g(K)$.

³A knot K^* in a rational homology sphere Y has *simple knot Floer homology* if $\text{rk } \widehat{\text{HFK}}(Y, K^*) = \text{rk } \widehat{\text{HF}}(Y)$. This definition does not require Y to be an L-space itself.

As mentioned in the paragraph following its statement, [Hed11, Theorem 3.1] (presented here in Theorem 14) applies to any L-space homology sphere, not just S^3 . Thus, since doubly primitive knots are surgery dual to one-bridge knots in lens spaces, [Hed11, Theorem 3.1] implies that such surgery duals are either simple knots or one of the Hedden knots T_L or T_R . In Section 5 we clarify and correct our work in [Bak14b] and further classify when Hedden's knots are dual to integral surgeries on knots homology spheres, \mathcal{P} in particular. Tange has produced a list of simple knots in lens spaces that are surgery dual to knots in \mathcal{P} [Tan09a], and it is expected that this list is complete. Verifying its completeness will affirm Conjecture 6.

Let us collect various properties of our knots.

Theorem 7. *Let $p = 3n^2 + n + 1$ and $q = -3n + 2$. Then the following hold for our family of knots K_n in \mathcal{P} :*

- (1) *Positive p -surgery on K_n produces a lens space $L(p, q)$.*
- (2) *K_n is fibered and supports the tight contact structure on \mathcal{P} .*
- (3) *$g(K_n) = (p + 1)/2$*
- (4) *$\text{rk HFK}(L(p, q), K_n^*) = p + 2$*
- (5) *Let T_n be the $(3n + 1, n)$ -torus knot in S^3 , and note that p -surgery on T_n also produces $L(p, q)$. The surgery dual to K_n is homologous to the surgery dual to T_n .*
- (6) *$\Delta_{K_n}(t) = \Delta_{T_n}(t) - (t^{(p-1)/2} + t^{-(p-1)/2}) + (t^{(p+1)/2} + t^{-(p+1)/2})$.*

Theorem 7(5) is given as Lemma 12. The remainder of Theorem 7 follows from assembling the works of Hedden [Hed11], Rasmussen [Ras07], and Tange [Tan11, Tan09b]. In fact, appealing to Greene's proof of the Lens Space Realization Problem [Gre13], we tease out the following general theorem from which Theorem 7(3)&(6) follow.

Theorem 8. *Suppose that p -surgery on knot K in an L-space homology sphere Y with $d(Y) = 2$ produces the lens space $L(p, q)$. Then $p = 2g(K) - 1$ if and only if p -surgery on some Berge knot B in S^3 also produces the lens space $L(p, q)$ in which the surgery duals to K and B are homologous.*

When this holds, $\Delta_K(t) = \Delta_B(t) - (t^{(p-1)/2} + t^{-(p-1)/2}) + (t^{(p+1)/2} + t^{-(p+1)/2})$.

2. QUESTIONS ABOUT KNOTS IN THE POINCARÉ HOMOLOGY SPHERE WITH LENS SPACE SURGERIES.

2.1. Homology classes of knots.

Question 9.

- (1) *For which Berge knots B that have a positive, odd p -surgery producing the lens space $L(p, q)$, is there a knot K in \mathcal{P} such that $2g(K) - 1 = p$, p -surgery produces the lens space $L(p, q)$, and the surgery duals B^* and K^* are homologous?*
- (2) *For a Berge knot B as above, can there be two distinct knots K_1 and K_2 in \mathcal{P} with p -surgery to $L(p, q)$ such that the surgery duals B^* , K_1^* , and K_2^* are all homologous?*

Presently the only Berge knots known to answer Question 9 (1) are the Berge knots of type VII (those that embed in the fiber of the trefoil) due to [Hed11, Bak14b] and the $(3n + 1, n)$ -torus knots with $p = 3n^2 + n + 1$ due to our knots introduced here. Indeed, what about the case of the $(3n + 1, n)$ -torus knots when $p = 3n^2 + n - 1$?

To clarify the word 'distinct' in Question 9 (2), note that the mapping class group of the Poincaré homology sphere is trivial [BO91, Théorém 3 & Corollaire 4]. Thus knots in \mathcal{P} that are related by a diffeomorphism of \mathcal{P} are also isotopic.

2.2. Symmetries. One may observe that, thus far, all the known examples of knots in \mathcal{P} with a lens space surgery are strongly invertible. Hence there is an involution of \mathcal{P} taking each of these knots with some orientation to its reverse. Wang-Zhou have shown that if a knot in S^3 other than a torus knots has a non-trivial lens space surgery, then its only possible symmetry is a strong involution [WZ92].

Question 10.

- (1) *Is every knot in \mathcal{P} with a lens space surgery strongly invertible?*
- (2) *What are the possible symmetries of a knot in \mathcal{P} with a lens space surgery?*

In the appendix to this article, Hoffman provides Theorem A.1 which answers Question 10(2): A hyperbolic knot in \mathcal{P} with a lens space surgery either has no symmetries or just a single strong involution.

2.3. Hopf plumbings. Let us inquire about another potential analogy with the Berge knots in S^3 . The Giroux Correspondence says that any two fibered knots supporting the same contact structure are related by a sequence of plumbings and de-plumbings of positive Hopf bands [Gir02]. Since the Berge knots (with positive surgeries to lens spaces) can all be expressed as closures of positive braids⁴, each Berge knot can be obtained from the unknot by a sequence of plumbings of Hopf bands. No de-plumbings are necessary.

In \mathcal{P} the unknot is not fibered, but the knot J that is -1 -surgery dual to the (negative) trefoil is. As we shall see in Lemma 13, J is a reasonable surrogate for the unknot in \mathcal{P} : it is the unique genus one fibered knot in \mathcal{P} , and it supports the tight contact structure on \mathcal{P} . Since a knot in \mathcal{P} with a lens space surgery is fibered and supports the tight contact structure (by Theorem 7(2)), the Giroux Correspondence says that it may be obtained from J by a sequence of plumbings and de-plumbings of positive Hopf bands.

Question 11. *If a knot in \mathcal{P} has a lens space surgery, then can it be obtained from J by a sequence of plumbings of Hopf bands?*

3. NOTATION AND CONVENTIONS

Let K be a knot in an oriented 3-manifold M . Choose an orientation on K and let μ be a meridian of K in the torus $\partial\mathcal{N}(K)$ that positively links K . Let λ be an oriented curve in $\partial\mathcal{N}(K)$ that is isotopic to K in $\mathcal{N}(K)$; if K is null-homologous in M , we choose λ so that it is null-homologous in $M - \mathcal{N}(K)$. If γ is an essential simple closed curve in $\partial\mathcal{N}(K)$, then when it is oriented $[\gamma] = p[\mu] + q[\lambda] \in H_1(\partial\mathcal{N}(K))$ where p, q are coprime integers; changing the orientation of γ changes the signs of both p and q . We refer to both the unoriented isotopy class of γ in $\partial\mathcal{N}(K)$ and the number $p/q \in \mathbb{Q} \cup \{\infty\}$ as a *slope* and (when λ is null-homologous) say it is *positive* if $0 < p/q < \infty$. If the slope is integral, we also say it is a *longitude* or a *framing* of K .

The result of γ -Dehn surgery on the knot $K \subset M$ is the manifold $M_\gamma(K)$ (or sometimes K_γ) obtained by attaching $S^1 \times D^2$ to $M - \mathcal{N}(K)$ along the torus $\partial\mathcal{N}(K)$ so that $\text{pt} \times \partial D^2$ is identified with the slope γ . The image of the curve $S^1 \times \text{pt}$ in $M_\gamma(K)$ is a knot called the *surgery dual* to K and denoted K^* . Observe that γ is the meridian of K^* and μ -surgery on K^* returns M with surgery dual $K = (K^*)^*$.

If M is the double branched cover of S^3 over a link L , an arc κ such that $\kappa \cap L = \partial\kappa$ lifts to a knot K in the cover M . Via the Montesinos trick [Mon75], each integral surgery on K corresponds to a *banding* along κ . With $I = [-1, 1]$, if $R = I \times I$ is a disk embedded in S^3 such that $R \cap L = \partial I \times I$ and $\{0\} \times I = \kappa$, then the link $L' = (L - \partial I \times I) \cup I \times \partial I$ is the result of a banding along κ . The arc $\kappa^* = I \times \{0\}$ is the dual arc of the banding. In the double branched cover over L' , κ^* lifts to the dual knot K^* of the corresponding integral surgery on K .

The lens space $L(p, q)$ is defined to be the manifold that results from $-p/q$ -surgery on the unknot in S^3 . The two-bridge link $B(p, q)$ is the link in S^3 whose double branched cover is $L(p, q)$. Using the continued fraction $[x_1, x_2, \dots, x_n] = x_1 - 1/(x_2 - 1/(\dots - 1/x_n))$ to express $-p/q$ describes the two-bridge link $B(p, q)$ geometrically in plat presentation as in the lower left of Figure 1.

A non-trivial knot in a lens space $L(p, q)$ is *simple* (or *grid number one*) if, in the standard genus one Heegaard diagram of the lens space, it may be represented by two of the p intersection points. The simple knot is then the union of the arcs connecting those points in the two meridional disks whose boundaries are described by the diagram. Equivalently, a simple knot (including the trivial knot) is a knot represented by a doubly pointed genus 1 Heegaard diagram of $L(p, q)$ that has p intersection points. There is one for each homology class $k \in H_1(L(p, q))$ in $L(p, q)$, denoted $K(p, q, k)$. See for example [Ras07, Hed11, BGH08].

4. PROOFS

Proof of Theorem 1. Figure 1 exhibits arcs κ_n on the pretzel knot $P(-2, 3, 5)$ that have a banding to a two-bridge link. By passing to the double branched covers (and using the Montesinos Trick), this describes knots K_n in \mathcal{P} that have an integral surgery to a lens space.

⁴This is remarked preceding the Conjecture of [GT00], for example.

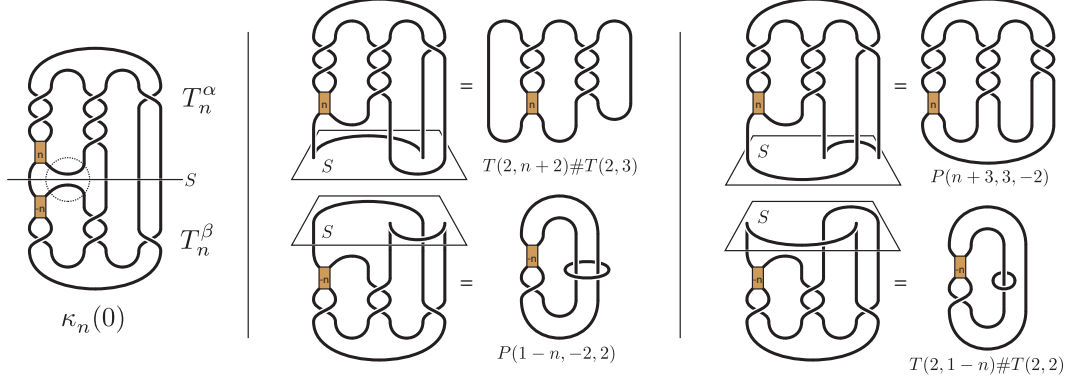


FIGURE 2. (Left) The bridge sphere S for $P(-2, 3, 5)$ splits $\kappa_n(0)$ into two tangles, T_n^α above and T_n^β below. (Middle) Filling T_n^α with a rational tangle to get $T(2, n+2)\#T(2, 3)$ makes T_n^β into $P(1-n, -2, 2)$. (Right) Filling T_n^β with a rational tangle to get $T(2, 1-n)\#T(2, 2)$ makes T_n^α into $P(n+3, 3, -2)$.

Since K_n lies in a genus 2 Heegaard surface, its tunnel number is at most 2. If K_n were to have tunnel number one, then every surgery on K_n would have Heegaard genus at most 2. However we will see that surgery along the framing induced by the Heegaard surface produces a toroidal manifold which, according to Kobayashi's classification [Kob84], does not have Heegaard genus 2. (The same scheme was recently employed in [EMJMM14] to demonstrate a family of strongly invertible knots in S^3 with a Seifert fibered surgery and tunnel number 2.)

Kobayashi shows that if $M = M^\alpha \cup_T M^\beta$ is a genus 2 manifold decomposed along a torus T into two atoroidal manifolds, then one of M^α or M^β admits a Seifert fibering over the disk with 2 or 3 exceptional fibers or over the Mobius band with up to 2 exceptional fibers such that filling the other along the slope induced by a regular fiber in T produces a lens space. (This is more general than Kobayashi's classification but suitable for our needs.) The slope of a regular fiber may be identified since filling one of these Seifert fibered spaces along the slope of a regular fiber produces a reducible manifold.

Figure 2(Left) shows that the result of a 0-framed banding along κ_n has a sphere S dividing it into two 2-string tangles. Up to homeomorphism, these two tangles are the tangle sums $T_n^\alpha = \frac{1}{n+2} + \frac{1}{3}$ and $T_n^\beta = \frac{1}{1-n} + \frac{1}{-2}$. Their corresponding double branched covers, M_n^α and M_n^β , are in general each Seifert fiber spaces over the disk with exactly two exceptional (and non-degenerate) fibers: M_n^α has type $D^2(|n+2|, 3)$ and M_n^β has type $D^2(|1-n|, 2)$. This fails for M_n^α when $n = -1, -2, -3$ and for M_n^β when $n = 0, 1, 2$; in each, the middle value yields a degenerate Seifert fibration while the other values yield solid tori. In these cases the torus T that is the double branched cover of S is compressible and so Kobayashi's classification does not apply; moreover, one may observe that K_n has tunnel number one in these cases. Hence we assume $n \notin \{-3, -2, -1, 0, 1, 2\}$.

When $n = -1$ or 3 , M_n^β has type $D^2(2, 2)$ and thus is a twisted I -bundle over the Klein bottle. Therefore it has an alternative Seifert fibration over the Mobius band with no exceptional fibers. We already assume $n \neq -1$. We shall assume $n \neq 3$ as well.

Figure 2(Middle) shows the results of filling T_n^α and T_n^β with the rational tangle ρ^α defined by a pair of arcs in S so that $T_n^\alpha(\rho^\alpha)$ is composite. We then observe that the filling $T_n^\beta(\rho^\alpha)$ is the pretzel link $P(1-n, 2, 2)$, and this is a two-bridge link if and only if $n = 0, 2$. We have already omitted these values of n .

Figure 2(Right) shows the results of filling T_n^α and T_n^β with the rational tangle ρ^β defined by a pair of arcs in S so that $T_n^\beta(\rho^\beta)$ is composite. We then observe that the filling $T_n^\alpha(\rho^\beta)$ is the pretzel link $P(n+2, 3, -2)$, and this is a two-bridge link if and only if $n = -1, -3$. We have already omitted these values of n too.

Therefore, to conclude that the knot K_n has tunnel number 2, it is sufficient to require that $n \notin \{-3, -2, -1, 0, 1, 2, 3\}$. \square

Lemma 12. *The knot K_n^* in $L(3n^2 + n + 1, -3n + 2)$ that is surgery dual to K_n is homologous to the knot T_n^* that is surgery dual to the $(3n + 1, n)$ -torus knot in S^3 .*

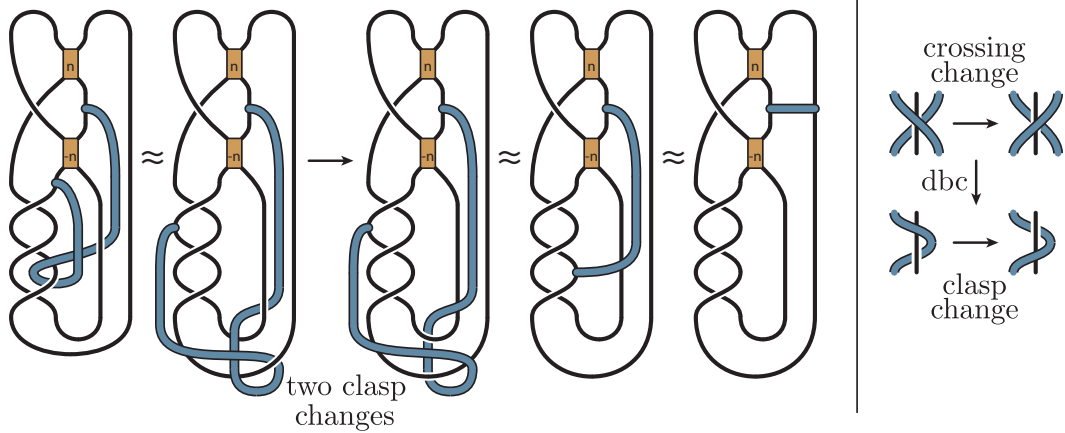


FIGURE 3. (Left) An isotopy of the arc κ_n^* on the two bridge link $B(3n^2 + n + 1, -3n + 2)$ from the lower left of Figure 1 has two clasp changes performed. The resulting arc τ_n^* is then isotoped to lie in a bridge sphere. (Right) A clasp change of an arc around a segment of a branch locus lifts to a crossing change about a segment of the fixed set in the double branched cover.

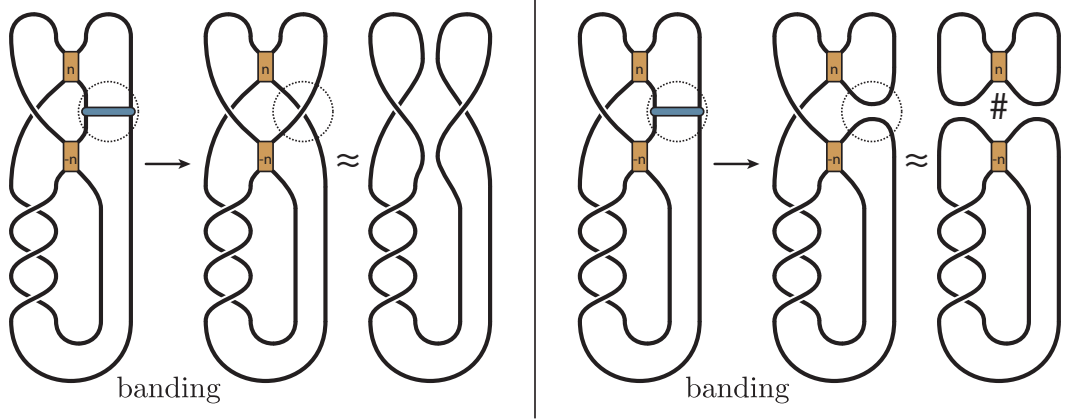


FIGURE 4. (Left) A banding of $B(3n^2 + n + 1, -3n + 2)$ along the arc τ_n^* produces the unknot. (Right) A different banding of $B(3n^2 + n + 1, -3n + 2)$ along the arc τ_n^* produces the connected sum of $B(n, -1)$ and $B(3n + 1, 3)$.

Proof. Starting from the lower left of Figure 1 where the dual arc κ_n^* is presented on a standard form of the two bridge link $B(3n^2 + n + 1, -3n + 2)$, Figure 3 (Left) shows how two clasp changes (and isotopy) transforms κ_n^* into an arc τ_n^* in a bridge sphere. Since τ_n^* lies in the bridge sphere, its lift to the double branched cover is a knot T_n^* in the Heegaard torus of the lens space. Lifting the transformation of Figure 3 (Left) to the double branched cover thus shows that K_n^* and T_n^* are related by two crossing changes. (The clasp changes lift to crossing changes as indicated in Figure 3(Right).) Hence these knots are homotopic, and therefore homologous, in the lens space.

Finally, Figure 3(Left) shows a banding of $B(3n^2 + n + 1, -3n + 2)$ along the arc τ_n^* to the unknot. Thus we may identify T_n^* as surgery dual to a torus knot T_n in S^3 . Because Figure 3(Left) shows a banding of $B(3n^2 + n + 1, -3n + 2)$ along the arc τ_n^* to the connected sum $B(n, -1) \# B(3n + 1, 3)$, the torus knot T_n has a reducible surgery to $L(n, -1) \# L(3n + 1, 3) \cong L(-n, 3n + 1) \# L(3n + 1, -n)$ which is the mirror of $L(n, 3n + 1) \# L(3n + 1, n)$. Thus T_n must be the $(3n + 1, n)$ -torus knot (for example see [Mos71]). \square

Proof of Theorem 7. (1) Theorem 1 shows that either $\pm p$ -surgery on K_n is a lens space. Tange shows that any integral lens space surgery on a knot in \mathcal{P} must be positive [Tan11, Theorem 1.1].

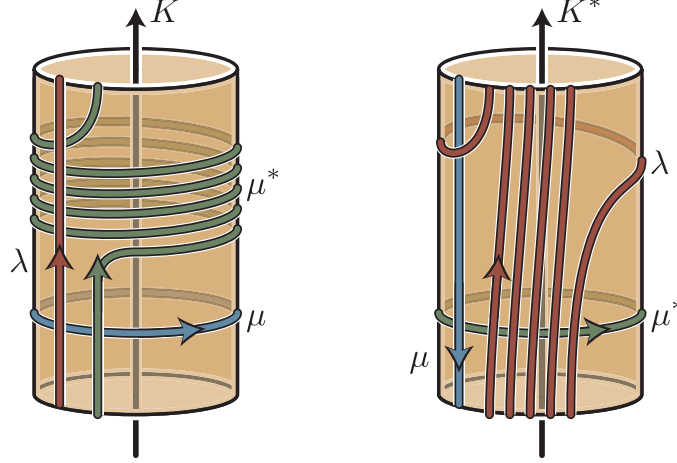


FIGURE 5. (Left) A regular neighborhood of an oriented knot K in a homology sphere Y has meridian μ linking K positively and longitude λ oriented as the boundary of a Seifert surface. The surgery curve μ^* of positive slope $5/1$ shown here is oriented as $[\mu^*] = [\lambda] + 5[\mu]$ in $H_1(\partial\mathcal{N}(K))$. (Right) Performing μ^* -surgery on K produces the rational homology sphere Z and surgery dual knot K^* . Orienting K^* to be linked positively by its meridian μ^* , the boundary of the (rational) Seifert surface λ is homologous to $5[K^*]$ in $\mathcal{N}(K^*)$ while the longitude μ is homologous to $-[K^*]$.

(2) In [Tan11, Theorem 3.1], the paragraph following, and its proof, Tange further shows that if an L-space homology sphere Y contains a knot K with irreducible exterior for which a positive integer surgery produces an L-space, then K is a fibered knot supporting a tight contact structure. (Tange notes that the fiberedness of K follows from the proof of [OS05, Theorem 1.2] and [Ni07]. Tange then demonstrates that the Heegaard Floer contact invariant of the contact structure supported by K is non-zero.)

(5) If T_n is the $(3n+1, n)$ -torus knot in S^3 , then p -surgery on T_n produces the lens space $L(p, q)$. By Lemma 12, the surgery dual knot T_n^* is homologous to the knot K_n^* that is surgery dual to K_n .

(3) & (6) Since the dual knots K_n^* and T_n^* are homologous, Theorem 8 implies $2g(K_n) = p-1$ and the stated relationship of the Alexander polynomials of K_n and T_n .

(4) Knowing that $g(K_n) = (p+1)/2$ implies $\text{rk } \widehat{\text{HFK}}(L(p, q), K_n^*) = p+2$ [Hed11, Theorem 1.4] (cf. [Ras07, Proposition 4.5]). \square

Proof of Theorem 8. Let K^* be the surgery dual to K in $L(p, q)$ and (for some choice of orientations) let J^* be the simple knot in $L(p, q)$ homologous to K^* . Then J^* has an integral surgery to an L-space homology sphere Y_J , [Ras07, Theorem 2].

Since *positive* p -surgery on K gives $L(p, q)$, then the self-linking number of K^* is $-1/p \pmod{1}$, see [Ras07, Section 2]. This is demonstrated in Figure 5: If Σ is an oriented Seifert surface for K giving the oriented longitude $\lambda = \partial\Sigma$, then the positively linking meridian μ of K is oriented as shown. In order for λ to run positively along K^* , we need $\mu^* \cdot \lambda > 0$. Thus we must orient μ^* so that $[\mu^*] = [\lambda] + p[\mu]$. This forces μ to be an anti-parallel longitude of K^* . Hence we may use $-\mu$ to calculate the self-linking number of K^* as $(-\mu \cdot \lambda)/p \pmod{1}$.

With this set-up, μ -surgery on K^* may be regarded as -1 -surgery on K^* . This means $Y = K_{-1}^*$ in the notation of [Ras07], though not explicitly stated. Similarly we also have $Y_J = J_{-1}^*$.

If $2g(K) - 1 \neq p$, then the knots J^* and K^* have isomorphic knot Floer homology [Ras07, Theorem 2] (cf. [Hed11, Theorem 1.4]). Having isomorphic (and simple) knot Floer homology would imply that the L-space homology spheres Y_J and Y have the same d -invariants. Hence $d(Y_J) = 2$ too. But now Y_J cannot be S^3 since $d(S^3) = 0$.

If $2g(K) - 1 = p$ then $\text{width } \widehat{\text{HFK}}(L(p, q), K^*) = 2p$ [Ni09], cf. [Ras07, Theorem 4.3]. Thus $d(Y) = d(Y_J) + 2$ by [Ras07, Proposition 5.4] and so $d(Y_J) = 0$. Greene's solution to the Lens Space Realization Problem

[Gre13] proceeds by first identifying the pairs (lens space, homology class) that contain knots for which some integral surgery produces an L-space homology sphere with $d = 0$, and then observing that each of these pairs contains the surgery dual to a Berge knot. Since the lens space surgery duals to Berge knots are simple knots, we have that in fact J^* is surgery dual to a Berge knot B and $Y_J = S^3$.

For the statement about Alexander polynomials, let us first summarize work of Tange on Alexander polynomials of knots in L-space homology spheres for which a positive integral surgery yields a lens space [Tan09b]. The symmetrized Alexander polynomial of a knot K in an L-space homology sphere with a non-trivial L-space surgery can be expressed as

$$\Delta_K(t) = \sum_{i \in \mathbb{Z}} a_i(K) t^i \in \mathbb{Z}[t^{\pm 1}] = (-1)^k + \sum_{j=1}^k (-1)^{k-j} (t^{n_j} + t^{-n_j})$$

following the arguments of [OS05]. For a given positive integer p , pass to the quotient $\mathbb{Z}[t^{\pm 1}]/(t^p - 1)$ to obtain a polynomial $\tilde{\Delta}_K(t) = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \tilde{a}_i(K) t^i$ with coefficients $\tilde{a}_i(K) = \sum_{j \equiv i \pmod{p}} a_j(K)$. Assuming that p -surgery gives a lens space Z , then $2g(K) - 1 \leq p$ [KMOS07]. Note that K is fibered⁵ and hence the degree of $\Delta_K(t)$ equals $2g(K)$. Because of this and that the coefficients of $\Delta_K(t)$ are $a_i(K) = 0$ or ± 1 (and the non-zero ones alternate in sign), it follows that $\tilde{a}_i(K) = 0, \pm 1$, or 2 where $\tilde{a}_i(K) = 2$ implies that both p is even and $i = p/2$ [Tan09b, Corollary 2]. Furthermore, one may determine that the coefficients $\tilde{a}_i(K)$ and the polynomial $\tilde{\Delta}_K(t)$ only depend on the lens space Z and the homology class of the surgery dual knot K^* [Tan09b, Theorem 5] (cf. [KY07]). In particular, if $p = 2g(K) - 1$ then

$$\Delta_K(t) = \tilde{\Delta}_K(t) - (t^{(p-1)/2} + t^{-(p-1)/2}) + (t^{(p+1)/2} + t^{-(p+1)/2})$$

where the indices for the coefficients of $\tilde{\Delta}_K(t)$ are taken to be the representatives of $\mathbb{Z}/p\mathbb{Z}$ from $-(p-1)/2$ and $(p-1)/2$. See also the discussion preceding [Tan09a, Proposition 3.3].

Now in our present situation, since the Berge knot B in S^3 has p -surgery to $L(p, q)$ in which the surgery dual is the simple knot J^* , we therefore have $2g(B) < p$ because p is odd [Hed11, Theorem 3.1]⁶. Hence $\tilde{a}_i(B) = a_i(B)$ for all $i \in \mathbb{Z}/p\mathbb{Z}$ and $\tilde{\Delta}_B(t) = \Delta_B(t)$. Because the surgery dual K^* is homologous to the simple knot B^* in $L(p, q)$, $\tilde{\Delta}_K(t) = \tilde{\Delta}_B(t)$. Therefore we have

$$\Delta_K(t) = \Delta_B(t) - (t^{(p-1)/2} + t^{-(p-1)/2}) + (t^{(p+1)/2} + t^{-(p+1)/2})$$

as claimed. □

Lemma 13. *There is a unique genus one fibered knot J in \mathcal{P} . As an open book, it supports the tight contact structure on \mathcal{P} . Furthermore, it is surgery dual to the negative trefoil knot in S^3 .*

Proof. This can be seen in the spirit of [Bak14a] which relates genus one fibered knot to axes of closed 3-braids as follows: Noting that since \mathcal{P} has a unique genus 2 Heegaard splitting [BO91], $P(-2, 3, 5)$ is the only 3-bridge link whose double branched cover gives \mathcal{P} . (In fact \mathcal{P} is the double branched cover of no other link.) Since $P(-2, 3, 5)$ is isotopic to the $(3, 5)$ -torus knot, its 3-braid axis A lifts to a genus one fibered knot $J \subset \mathcal{P}$. By [BM93, The Classification Theorem] and [Bak14a, Lemma 3.8], for instance, A is the only 3-braid axis for $P(-2, 3, 5)$ up to isotopy of unoriented links. Hence J is the only genus one fibered knot up to homeomorphism in \mathcal{P} . Since the mapping class group of \mathcal{P} is trivial [BO91], J is the only genus one fibered knot in \mathcal{P} up to isotopy.

In the double branched cover, a braid presentation of the branch locus lifts to a Dehn twist presentation of the monodromy of the lift of the axis. Since the axis A presents $P(-2, 3, 5)$ as a positive braid (indeed, as the $(3, 5)$ -torus knot), we obtain a presentation of the monodromy of J as a product of positive Dehn twists. Hence it supports the tight contact structure on \mathcal{P} [Gir02].

One may view J as surgery dual to the negative trefoil in several ways. Taking the route through branched covers, this is demonstrated in [Bak14b, Proof 2], though for the mirrored situation. □

⁵Since lens spaces are irreducible, a knot in a homology sphere with a lens space surgery must have irreducible exterior. Thus K has irreducible exterior and so it is fibered by [OS05, Theorem 1.2] and [Ni07]; cf. Theorem 7(2).

⁶See also [Gre13, Theorem 1.4].

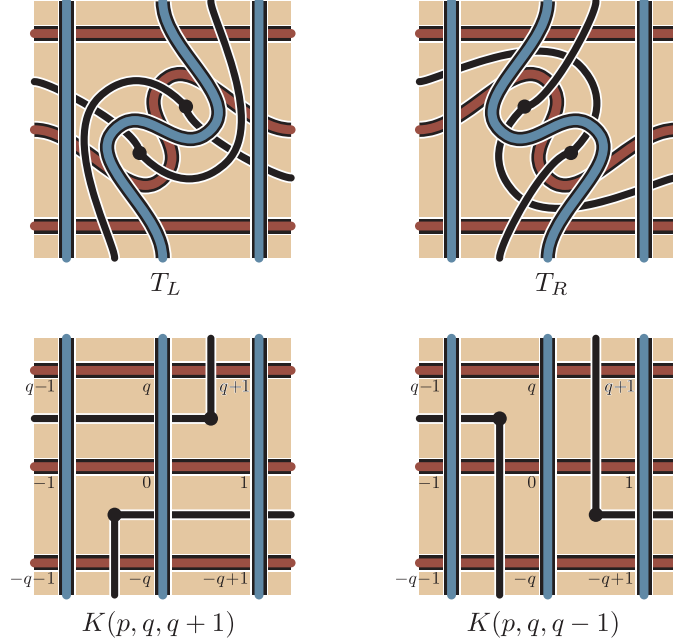


FIGURE 6. Local portions of the doubly pointed diagrams for the almost-simple knots T_L and T_R and their associated simple knots $K(p, q, q+1)$ and $K(p, q, q-1)$; cf. [Bak14b, Figure 1] and [Hed11, Figure 3]. In the bottom two pictures, the intersection points of the red α -curve and blue β -curve are numbered in order mod p along the α -curve.

5. HEDDEN'S ALMOST-SIMPLE KNOTS.

In each lens space $L(p, q)$ with coprime $p > q > 0$, Hedden diagrammatically describes two unoriented 1-bridge knots T_L and T_R (*the Hedden knots*) via the doubly-pointed genus 1 Heegaard diagrams of $L(p, q)$ with $p+2$ intersection points [Hed11, Figure 3]. An alternative presentation of local pictures of these diagrams near the two marked points are shown in the top row of Figure 6. Since the *simple knots* in $L(p, q)$ are those that can be described via doubly-pointed genus 1 Heegaard diagrams of $L(p, q)$ with only p intersection points, we like to regard the Hedden knots as *almost-simple*. The bottom row of Figure 6 shows the same portion of the diagram for related simple knots. These knots are notable because of the following theorem.

Theorem 14 ([Hed11, Theorem 3.1]). *If K^* is a 1-bridge knot in $L(p, q)$ with an integral surgery to an L-space homology sphere, then either*

- $p > 2g - 1$ and K^* is a simple knot, or
- $p = 2g - 1$ and K^* is either T_L or T_R .

where g is the Seifert genus of the surgery dual knot.

Though the statement of this theorem in [Hed11] is explicitly only for surgeries to S^3 , Hedden notes that his proof applies for surgeries to any L-space homology sphere.

Here we would like to clarify a few points about these almost-simple knots, correct a couple of misstatements in [Bak14b], and discuss the homology spheres that may be obtained by integral surgery on them following the arguments of [Bak14b]. Afterwards, we summarize these results in Proposition 15.

The knots T_L and T_R are not homologous in general, contrary to what was stated in [Bak14b]. One can observe this directly from the descriptions in Figure 6 (or [Hed11, Figure 3]), noting that the magnitude of their algebraic intersection numbers with the blue β -curve differ by 2. One may further observe that the mirror of the knot T_L in $L(p, q)$ is the knot T_R in $L(p, -q)$.

Furthermore, the proof in [Bak14b] that T_L differs from (and thus is homologous to) $K(p, q, q+1)$ by a crossing change also shows that T_R differs from $K(p, q, q-1)$ by a crossing change. Unfortunately, [Bak14b,

Figure 2] should be mirrored through the Heegaard torus so that the red α -disks are below, the blue β -disks are above, and the twist in T_L is right-handed in order to be consistent with the orientation conventions. In particular, the crossing change from $K(p, q, q+1)$ to T_L is achieved by a -1 -surgery on a loop C_L in the Heegaard that encircles the two base-points in the diagram for $K(p, q, q+1)$ of bottom left Figure 6. The crossing change from $K(p, q, q-1)$ to T_R may be similarly achieved by a $+1$ -surgery on a loop C_R .

A knot homologous to $K(p, q, k)$ has an integral surgery to a homology sphere if and only if its self linking is $\pm 1/p \pmod{1}$, or equivalently if and only if $k^2 = \pm q \pmod{p}$ [Ras07, Lemma 2.6], cf. [FS80]. Here the choice of sign of \pm is consistent and agrees with whether this integral surgery is a ± 1 -surgery; cf. Figure 5 and the discussion in the proof of Theorem 8.

If T_L in $L(p, q)$ is dual to positive p -surgery on a knot K in a homology sphere Y , then so is $K(p, q, q+1)$ and hence $(q+1)^2 = -q \pmod{p}$. Making the substitution $-k = q+1$ gives the equation $k^2 + k + 1 = 0 \pmod{p}$. Thus $K(p, q, q+1)$ is dual to a type VII Berge knot B , one that lies in the fiber of a trefoil knot; see [Ras07, Section 6.2] and [Gre13, Section 1.2]. Since its fiber positively frames B , this trefoil knot is the *negative* trefoil (not the positive trefoil as stated in [Bak14b]), and it can be identified with C_L under the S^3 surgery on $K(p, q, q+1)$. Performing -1 -surgery on the negative trefoil C_L in S^3 produces \mathcal{P} and takes $B \subset S^3$ to $K \subset \mathcal{P}$. Since the linking of B and C_L is 0, the positive p -surgery on B becomes the positive p -surgery on K . Consequently, if -1 -surgery on T_L is a homology sphere, it is \mathcal{P} .

Similarly, if T_R in $L(p, q)$ is dual to positive p -surgery on a knot K' in a homology sphere, then $(q-1)^2 = -q \pmod{p}$. Making the substitution $k = q-1$ gives the equation $k^2 - k - 1 = 0 \pmod{p}$. Thus $K(p, q, q-1)$ is dual to a type VIII Berge knot B' , one that lies in the fiber of the figure eight knot. This figure eight knot may be identified with C_R . Performing $+1$ -surgery on C_R in S^3 produces the Brieskorn sphere $\Sigma(2, 3, 7)$ taking the knot B' to the knot K' . Consequently, if -1 -surgery on T_R is a homology sphere, it is $\Sigma(2, 3, 7)$.

This coincides with the difference between the τ -invariants of these knots as noted by Rasmussen in the last two paragraphs of [Ras07, Section 5]: $\tau(T_L, \mathfrak{s}_0) = -1$ and so $\tau(T_R, \mathfrak{s}_0) = +1$. Rasmussen further shows that if integral surgery on T_L or T_R produces a homology sphere, then it is an L-space homology sphere if and only if the surgery is a -1 -surgery on T_L or a $+1$ -surgery on T_R [Ras07, Proposition 4.5].

In summary, the above discussion shows.

Proposition 15.

- (1) In $L(p, q)$, T_L is homologous to the simple knot $K(p, q, q+1)$ and T_R is homologous to the simple knot $K(p, q, q-1)$.
- (2) The mirror of $(L(p, q), T_L)$ is $(L(p, -q), T_R)$.
- (3) If $(T_L)_{-1}$ is a homology sphere, then it is $\mathcal{P} = \Sigma(2, 3, 5)$ and $K(p, q, q+1)$ is positive surgery dual to a type VII Berge knot.
- (4) If $(T_R)_{-1}$ is a homology sphere, then it is $\Sigma(2, 3, 7)$ and $K(p, q, q-1)$ is positive surgery dual to a type VIII Berge knot.

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APPENDIX A. SYMMETRIES OF KNOTS WITH LENS SPACE AND POINCARÉ HOMOLOGY SPHERE SURGERIES

NEIL R. HOFFMAN

The aim of this appendix is to compute the symmetry group of a hyperbolic knot in a lens space with a surgery to the Poincaré homology sphere \mathcal{P} . Specifically, it will provide a proof of the following theorem:

Theorem A.1. *Let M be a one-cusped hyperbolic manifold admitting a Poincaré homology sphere filling and a lens space filling. Then, the symmetry group of M is trivial or generated by a single strong inversion.*

Throughout, we will use the notation for $(Q, K, -)$ to denote the complement of a (framed, embedded) knot K in a manifold (or orbifold) Q and (Q, K, α) to denote the filling of $(Q, K, -)$ along the peripheral curve α . Given an embedded link L , we will define $(Q, L, -)$ similarly, while pointing out that the embedding of L provides both a framing of L and an ordering of the components of L , and so if L in an n -component link, then we will denote by $(Q, L, (\alpha_1, \dots, \alpha_n))$ the filling of along the curves $\{\alpha_i\}$. Note, we use $\alpha_i = -$ to denote an unfilled cusp. Given two peripheral curves α and β we say the *distance* between α and β , $\Delta(\alpha, \beta)$ is the minimal (unoriented) geometric intersection number between the two curves. Finally, given an orbifold Q , the *singular locus* of Q is the set of points fixed by some non-trivial element of $\pi_1^{orb}(Q)$.

Using $Sym(M)$ to denote the symmetry group of a manifold and $Sym^+(M)$ to denote the orientation preserving symmetry group of M , the argument in this appendix will gradually “whittle-down” $Sym(\Sigma, K, -)$, the symmetry group of a hyperbolic knot complement in an integral homology sphere admitting a non-trivial lens space surgery. First, we give an argument which eliminates orientation reversing symmetries.

Proposition A.2. *If $(\Sigma, K, -)$ is knot complement in an integral homology sphere Σ and $(\Sigma, K, -)$, admits an $L(p, q)$ surgery with $p \geq 2$ and is not Seifert fibered, then $Sym(\Sigma, K, -) = Sym^+(\Sigma, K, -)$.*

Proof. Assume $(\Sigma, K, -)$ admits an orientation reversing symmetry τ , λ is the homologically determined longitude of $(\Sigma, K, -)$. Then $\tau(\lambda) = \pm\lambda$ (as a curve in $H_1((\Sigma, K, -), \mathbb{Z})$). Also, there must be a $\mu \in H_1(\partial(\Sigma, K, -), \mathbb{Z})$ which maps to a generator of $H_1((\Sigma, K, -), \mathbb{Z})$ such that (μ, λ) is a basis for $H_1(\partial(\Sigma, K, -), \mathbb{Z})$. Given these conditions $\tau(\mu) = \pm\mu$ and because τ is orientation reversing, then either $\tau(\lambda) = \pm\lambda$ or $\tau(\mu) = \pm\mu$ but not both.

If s is the slope of the $L(p, q)$ surgery, then, $s = p\mu + q'\lambda$ where $q' \in \mathbb{Z} - \{0\}$. In particular, s is not a generator of $H_1((\Sigma, K, -), \mathbb{Z})$. Therefore, s is not fixed by τ , in fact $\tau(s) = \pm(p\mu - q'\lambda)$. Furthermore, the distance between $\Delta(s, \tau(s)) = 2pq'$. Because $p \geq 2$, $2pq'$ is at least two. However, by the symmetry $\pi_1(\Sigma, K, s)$ and $\pi_1(\Sigma, K, \tau(s))$ are both cyclic. Since $(\Sigma, K, -)$ is not Seifert fibered, this contradicts the Cyclic Surgery Theorem [CGLS87]. \square

We state a lemma of Neumann and Reid established by the proof of [NR92, Proposition 9.1], which restricts the orientation preserving symmetry groups of knot complements in rational homology spheres generally.

Lemma A.3. [NR92, Proposition 9.1] *If $(M, K, -)$ is a hyperbolic knot complement in a rational homology sphere M , then $Sym^+(M, K, -)$ is cyclic or $\mathbb{Z}/2\mathbb{Z}$ extended by a cyclic group.*

Proposition A.2 shows that we should focus on orientation preserving symmetries. For the remainder of this note, we will assume that $Sym^+(M, K, -) = Sym(M, K, -)$ for a knot complement in an integral homology sphere M that admits a lens space filling. We can refine this notion to say that elements of $Sym(M, K, -)$ that reverse orientation of the homologically determined longitude are *strong inversions* and the remaining symmetries are in a cyclic subgroup $Z(M, K, -)$. This group $Z(M, K, -)$ was studied in Boyer, Boileau, Cebanu and Walsh [BBCW12] in the case that of a hyperbolic knot complement in a lens space that is an integral homology solid torus. The following lemma is an application of the ideas of Gonzalez-Acuña and Whitten [GAW92, Proof of Theorem 3.4] and the classification of cyclic quotients of S^3 given by Boileau, Boyer, Cebanu and Walsh [BBCW12, §3].

Lemma A.4. *If $(L(p, q), K, -)$ is a hyperbolic knot complement in a lens space $L(p, q)$ such that $H_1(L(p, q), K, -) = \mathbb{Z}$, then M is covered by $(S^3, K', -)$ for some knot K' in S^3 . Furthermore, $M/Z(L(p, q), K, -)$ is complement of a knot in an orbi-lens space.*

In light of the above lemma which is based on a classification of cyclic quotients of S^3 , it will also prove useful to have a list of manifolds and orbifolds that are cyclic quotients of \mathcal{P} . This lemma essentially follows from combining the classification of elliptic manifolds with the corresponding classification of elliptic orbifolds in Dunbar [Dun88].

Lemma A.5. *If $O_{\mathcal{P}}$ is an orbifold which is the cyclic quotient of \mathcal{P} by isometries, then $O_{\mathcal{P}}$ is homeomorphic to one of the following:*

- (1) A manifold $M_{\mathcal{P}}$ with $\pi_1(M_{\mathcal{P}}) \cong \pi_1(\mathcal{P}) \times \mathbb{Z}/n\mathbb{Z}$, with $(n, 30) = 1$,
- (2) an orbifold that fibers over $S^2(2, 3, 5)$ where the fixed point set of $O_{\mathcal{P}}$ has 1, 2, or 3 components, or

(3) an orbifold with base space S^3 and singular set a knot or link as pictured in Figure 7.

Moreover, in the case that $O\mathcal{P}$ is an orbifold, the components of the singular set are fixed locally by finite cyclic groups of relatively prime orders.

Proof. Case 1 follows directly from the classification of elliptic manifolds (see for example [Thu97, Theorem 4.4.14]), while Cases 2 and 3 follow from a careful reading of [Dun88]. In [Dun88], Dunbar also classifies orbifolds covered by elliptic manifolds where the singular loci are trivalent graphs. However, none of this type of orbifold can be cyclically covered by a manifold because the group of isometries that fixes a vertex in the trivalent graph is never cyclic. \square

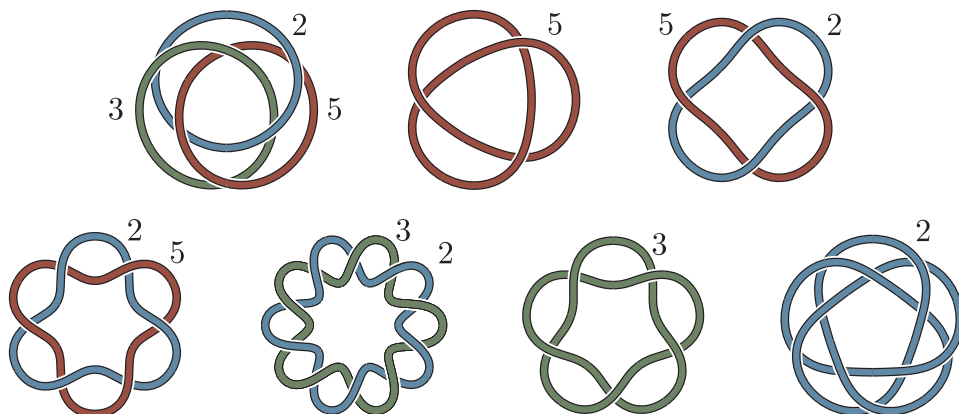


FIGURE 7. The possible singular loci of orbifolds cyclically covered by \mathcal{P} with underlying space S^3 . These are the relevant cases from a more general classification in [Dun88]. Following the notation of that paper, a link component labeled by n indicates a cone angle of $2\pi/n$ along that component in the singular set of the corresponding orbifold.

A simple case analysis shows that drilling the singular loci of orbifolds from the previous lemma results in several interesting manifolds.

Lemma A.6. *In Cases 2 and 3 of Lemma A.5:*

- If the singular set has one component, the complement of the singular set is a Seifert fibered space over the disk with two exceptional fibers.
- If the singular set has two components, the complement of the singular set is a Seifert fibered space over the annulus with one exceptional fiber.
- If the singular set has three components, the complement of the singular set is $F \times S^1$, where F is a pair of pants.

Lemma A.7. *If M is the complement of a hyperbolic knot in \mathcal{P} admitting a lens space surgery, then every (non-trivial) element of $Z(M, K, -)$ acts non-freely on M .*

Proof. Assume r_1 is the meridian of the knot in \mathcal{P} and r_2 is the slope corresponding to the lens space filling.

If $Z_f(M, K, -) \subset Z(M, K, -)$ is the subgroup that acts freely on M , then $M/Z_f(M, K, -)$ is a hyperbolic manifold with a torus cusp. Now assume $Z_f(M, K, -)$ is non-trivial. For $i = 1, 2$, let \bar{r}_i be the image of r_i in the quotient. Immediately, we will have three cases, depending on whether \bar{r}_1 and \bar{r}_2 remain primitive curves.

Case 1: \bar{r}_1 and \bar{r}_2 are primitive. Observe that $\Delta(\bar{r}_1, \bar{r}_2) = |Z_f(M, K, -)|\Delta(r_1, r_2)$ where $\Delta(\bar{r}_1, \bar{r}_2)$ is the distance in the boundary of the quotient and $\Delta(r_1, r_2)$ is measured in the boundary of $(M, K, -)$.

Since r_1 and r_2 have primitive images in the quotient manifold $M/Z_f(M, K, -)$, $Z_f(M, K, -)$ acts freely on the fillings (M, K, r_1) and (M, K, r_2) . Hence $M/Z_f(M, K, -)(\bar{r}_1) = (M, K, r_1)/Z_f(M, K, -)$ is a quotient of \mathcal{P} and $M/Z_f(M, K, -)(\bar{r}_2) = (M, K, r_2)/Z_f(M, K, -)$ is a quotient of a lens space, and thus they are both covered by S^3 . Therefore they are both finite manifolds.

The group $Z_f(M, K, -)$ acts freely on $\mathcal{P} = (M, K, r_1)$ and so $\mathcal{P}/Z_f(M, K, -)$ is an elliptic manifold. Since $\mathcal{P}/Z_f(M, K, -)$ is covered by \mathcal{P} , $|Z_f(M, K, -)|$ must be relatively prime to $|\pi_1(\mathcal{P})| = 120$. Thus, we must have $|Z_f(M, K, -)| > 6$. Therefore $\Delta(\bar{r}_1, \bar{r}_2) > 6$ which contradicts the bound on distance between two finite surgeries provided by Boyer and Zhang [BZ96].

Case 2: \bar{r}_1 is not primitive and \bar{r}_2 is primitive. In this case, we can exploit Lemma A.6, to see that the complement of the surgered torus would be Seifert fibered, which contradicts the fact that $M/Z_f(M, K, -)$ is hyperbolic.

Case 3: \bar{r}_2 is not primitive. Since $M/Z_f(M, K, -)(\bar{r}_2)$ is finite cyclic, it is either a lens space or an orbifold lens space. In the first case, \bar{r}_2 would be primitive while in the second the complement of the singular set would be the unknot. However, the singular set is the core of the surgery torus, but $M/Z_f(M, K, -)$ is hyperbolic. Thus, it cannot be a solid torus. \square

The final lemma needed to establish the theorem can be proved in a similar manner to the lemma above.

Lemma A.8. *If M is a hyperbolic knot complement in \mathcal{P} admitting a lens space surgery, then $|Z(M, K, -)| = 1$.*

Proof. By [BBCW12], $M/Z(M, K, -)$ is homeomorphic to the complement of a knot in an orbifold lens space where the fixed point set is one or two of the cores of a genus 1 (orbi-)Heegaard splitting of the orbifold lens space $L(p, q, a, b)$ (i.e. a decomposition into two solid orbifold-tori). Thus, $\mathcal{P}/Z(M, K, -)$ has a singular locus consisting of one, two, or three embedded circles.

Let $C(\mathcal{P}/Z(M, K, -))$ count the number of circles in the fixed point set of $\mathcal{P}/Z(M, K, -)$ and $C(L(p, q, a, b))$ count the number of circles in the fixed point set of $L(p, q, a, b)$. Then

$$C(\mathcal{P}/Z(M, K, -)) = C(L(p, q, a, b)) + e$$

where $e = 0, 1$. Also, just as above let r_1 be the slope of the \mathcal{P} surgery on M and r_2 be the slope of the lens space surgery.

Case 1: ($e = 1$). If $e = 1$, then r_1 is not a primitive curve in $M/Z(M, K, -)$. Let (a, c) or (a, b, c) be the orders of cyclic groups fixing the components of $\mathcal{P}/Z(M, K, -)$ where c corresponds to the order of the component corresponding to the core of surgered torus. By Lemma A.5, c is relatively prime to a and b . Thus, $Z(M, K, -)$ has a cyclic subgroup that acts freely, which contradicts Lemma A.7.

Case 2: ($e = 0$ and $C(\mathcal{P}/Z(M, K, -)) = C(L(p, q, a, b)) = 1$). In the first case, either the first point set of $\mathcal{P}/Z(M, K, -)$ corresponds to an exceptional fiber of $\mathcal{P}/Z(M, K, -)$ or $\mathcal{P}/Z(M, K, -)$ is an orbifold with base space S^3 with a singular locus corresponding $(5, 3)$ torus knot marked by cone angle π . In either case, we can drill out the singular locus K' from $L(p, q, a, b)$, resulting a solid torus (or $M/Z(M, K, r_2) - K'$ is a solid torus). Let K be the corresponding curve to K' in M (notice that this is disjoint from the cusp) and let K'' be the dual of K' in $\mathcal{P}/Z(M, K, -)$. Then by Lemma A.6, $\mathcal{P}/Z(M, K, r_1) - K''$ is a Seifert fibered space over the disk with two exceptional fibers. Therefore a one parameter family of surgeries s_n results in $M/Z(M, K, -) - K''(s_n)$. where s_n is chosen to be the family of lens space fillings for the Seifert fibered space over the disk with two exceptional fibers. Thus, $M/Z(M, K, -) - K''(s_n)$ admits two lens space fillings. Since the curve we drilled out from $M/Z(M, K, -)$ was a geodesic, $M/Z(M, K, -) - K''$ is hyperbolic [Koj88, Sak91], for all but at most finitely many choices of s_n , $M/Z(M, K, -) - K''(s_n)$ is hyperbolic. Therefore, by [CGLS87], (r_1, r_2) are distance one in $M/Z(M, K, -)$. Since (r_1, r_2) are distance $|Z(M, K, -)| \cdot d$ in M where d is the distance between r_1 and r_2 in ∂M , $Z(M, K, -)$ is trivial.

Case 3: ($C(\mathcal{P}/Z(M, K, -)) = C(L(p, q, a, b)) = 2$). This case is nearly identical to the previous case, except the singular locus of $M/Z(M, K, -)$ corresponds to 2 exceptional fibers. \square

It may be tempting to try to use $|Z(M, K, -)| \cdot d$ in the previous lemma place try and improve the bound in [BZ96] that $d \leq 2$. However, this bound is known to be sharp. Also, when $Z(M)$ is trivial, we lose the ability to “match up” the cores of the solid tori coming from the Heegaard splitting of the lens space with the exceptional fibers in \mathcal{P} , and so the arguments in this appendix will not apply.

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